POW#4 Solution: Newton’s second law tell us that torque equals moment of inertia times angular acceleration: \( mg\ell \sin \theta = m\ell^2 \ddot{\theta} \). With the approximation \( \sin \theta \approx \theta \), we have \( \ddot{\theta} = (g/\ell)\theta \). The general solution of this differential equation is \( \theta(t) \approx Ae^{\sqrt{g/\ell}t} + Be^{-\sqrt{g/\ell}t} \) where \( A \) and \( B \) are constants. Applying the initial conditions \( \theta(0) = \theta_0 \) and \( \dot{\theta}(0) = \omega_0 \), we have

\[
\theta(t) \approx \left( \frac{\theta_0 + \sqrt{\ell/g}\omega_0}{2} \right) e^{\sqrt{g/\ell}t} + \left( \frac{\theta_0 - \sqrt{\ell/g}\omega_0}{2} \right) e^{-\sqrt{g/\ell}t} \tag{1}
\]

The Heisenberg uncertainty principle tells us that the product of the initial horizontal displacement \( \ell \theta_0 \) and the initial momentum \( m\ell \omega_0 \) must be greater than, or approximately equal to, \( \hbar \). Thus, \( \theta_0 \omega_0 \gtrsim \hbar/(m\ell^2) \).

Since we are looking for a maximum time \( t \), and \( \sqrt{g/\ell} \approx 10.0 \), let’s ignore the term in Eq. (1) proportional to \( e^{-\sqrt{g/\ell}t} \). (As we will see, the maximum time \( t \) is relatively large, so this assumption is justified.)

Inserting the smallest possible \( \omega_0 \) (which will give the largest possible time for \( \theta(t) \) to reach values of order 1), we have

\[
\theta(t) \approx \frac{1}{2} \left( \theta_0 + \sqrt{\ell/g} \frac{\hbar}{m\ell^2 \theta_0} \right) e^{\sqrt{g/\ell}t} \tag{2}
\]

The maximum time \( t \) occurs when the coefficient of the exponential is a minimum. This occurs at \( \theta_0 = [\sqrt{\ell/g}\hbar/(m\ell^2)]^{1/2} \), and gives

\[
\theta(t) \approx \left[ \sqrt{\frac{\ell}{g}} \frac{\hbar}{m\ell^2} \right]^{1/2} e^{\sqrt{g/\ell}t} \tag{3}
\]
With $\theta(t) \approx 1$ we have

$$t \approx \frac{1}{2} \sqrt{\frac{\ell}{g}} \ln \left[ \sqrt{\frac{g m \ell^2}{\ell h}} \right]$$  \hspace{1cm} (4)

For $\ell = 0.1$, $m = 0.01$, $g = 9.8$ and $h = 1.05 \times 10^{-34}$, this yields $t \approx 3.6$ sec. Amazing!