(Super) Fluid Transport in the Unitary Fermi Gas

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Hydrodynamics

Hydrodynamics (undergraduate version): Newton’s law for continuous, deformable media.
Fluids: Gases, liquids, plasmas, …

Hydrodynamics (postmodern): Effective theory of non-equilibrium long-wavelength, low-frequency dynamics of any many-body system.

\[ \tau \sim \tau_{\text{micro}} \quad \text{and} \quad \tau \sim \lambda \]

\[ \tau \gg \tau_{\text{micro}}: \text{ Dynamics of conserved charges.} \]

Water: \((\rho, \epsilon, \vec{p})\)
Effective theories for fluids (Unitary Fermi Gas, $T > T_F$)

\[ \mathcal{L} = \psi^\dagger \left( i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2 \]

\[ \frac{\partial f_p}{\partial t} + \vec{v} \cdot \nabla_x f_p = C[f_p] \quad \omega < T \]

\[ \frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad \omega < T \frac{s}{\eta} \]

Strongly coupled: $\eta/s \sim 1$
Gradient expansion (simple non-relativistic fluid)

Simple fluid: Conservation laws for mass, energy, momentum

$$ \frac{\partial \rho}{\partial t} + \vec{\nabla} j^\rho = 0 $$

$$ \frac{\partial \epsilon}{\partial t} + \vec{\nabla} j^\epsilon = 0 $$

$$ \frac{\partial \pi_i}{\partial t} + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 $$

Ward identity: mass current = momentum density

$$ \vec{j}^\rho \equiv \rho \vec{v} = \vec{\pi} $$

Constitutive relations: Gradient expansion for currents

Energy momentum tensor

$$ \Pi_{ij} = P \delta_{ij} + \rho v_i v_j + \eta \left( \partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right) + O(\partial^2) $$
Gradient expansion, Kubo formula

Consider background metric $g_{ij}(t, x) = \delta_{ij} + h_{ij}(t, x)$. Linear response

$$\delta \Pi_{xy} = -\frac{1}{2} G_{R}^{xyxy} h_{xy}$$

Harmonic perturbation $h_{xy} = h_{0} e^{-i\omega t}$

$$G_{R}^{xyxy} = P - i\eta \omega + \ldots$$

Kubo relation: $\eta = - \lim_{\omega \to 0} \left[ \frac{1}{\omega} \text{Im} G_{R}^{xyxy}(\omega, 0) \right]$

Gradient expansion: $\omega \leq \frac{P}{\eta} \simeq \frac{s}{\eta} T.$
Superfluid hydrodynamics

Spontaneous symmetry breaking: \( \langle \Psi \rangle = v_0 e^{i\theta} \).

Goldstone boson is a new hydro mode: \( \vec{v}_s = \frac{\hbar}{m} \vec{\nabla} \theta \)

\[
\partial_t \vec{v}_s + \frac{1}{2} \vec{\nabla} (v_s^2) = -\vec{\nabla} \mu
\]

Momentum density: \( \pi_i = \rho_n v_{n,i} + \rho_s v_{s,i} \)

\[
\rho = \rho_n + \rho_s \quad \rho_s = \frac{1}{2} \frac{\partial F}{\partial w^2} \quad \vec{w} = \vec{v}_n - \vec{v}_s
\]

Stress tensor and energy current

\[
\Pi_{ij} = P \delta_{ij} + \rho_n v_{n,i} v_{n,j} + \rho_s v_{s,i} v_{s,j}
\]

\[
\vec{j}^e = sT \vec{v}_n + \left( \mu + \frac{1}{2} v_s^2 \right) \vec{\pi} + \rho_n \vec{v}_n \vec{v}_n \cdot \vec{w}
\]
Superfluid hydrodynamics

Dissipative stresses

\[ \delta \Pi_{ij} = -\eta \left( \nabla_i v_{n,j} + \nabla_j v_{n,i} - \frac{2}{3} \delta_{ij} \vec{\nabla} \cdot \vec{v}_n \right) \]

\[ - \delta_{ij} \left( \zeta_1 \vec{\nabla} \left( \rho_s (\vec{v}_s - \vec{v}_n) \right) + \zeta_2 \left( \vec{\nabla} \cdot \vec{v}_n \right) \right) \]

Equation of motions for \( v_s \):

\[ \dot{v}_s + \frac{1}{2} \nabla (v_s^2) = -\nabla (\mu + H) \]

with

\[ H = -\zeta_3 \vec{\nabla} \left( \rho_s (\vec{v}_s - \vec{v}_n) \right) - \zeta_4 \vec{\nabla} \cdot \vec{v}_n \]

Conformal symmetry:

\[ \zeta_1 = \zeta_2 = \zeta_4 = 0 \]

Son (2007)
Superfluid Hydrodynamics: Second Sound

1st (top) 2nd sound (bottom) in unitary Fermi gas

Superfluid mass fraction
CAG, He, BEC (th)

Grimm et al. (2013)
Fermi gas at unitarity: Field Theory

Non-relativistic fermions at low momentum

\[ \mathcal{L}_{\text{eff}} = \psi^\dagger \left( i \partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2 \]

Unitary limit: \( a \to \infty, \sigma \to 4\pi/k^2 \) \( (C_0 \to \infty) \)

This limit is smooth (HS-trafo, \( \Psi = (\psi_\uparrow, \psi_\downarrow^\dagger) \))

\[ \mathcal{L} = \Psi^\dagger \left[ i \partial_0 + \sigma_3 \frac{\nabla^2}{2m} \right] \Psi + (\Psi^\dagger \sigma_+ \Psi \phi + \text{h.c.}) - \frac{1}{C_0} \phi^* \phi , \]

Low \( T \) \( (T < T_c \sim \mu) \): Pairing and superfluidity
**Low T: Phonons** Goldstone boson $\psi\psi = e^{2i\varphi} \langle \psi\psi \rangle$

$$\mathcal{L} = c_0 m^{3/2} \left( \mu - \dot{\varphi} - \frac{(\vec{\nabla} \varphi)^2}{2m} \right)^{5/2} + \ldots$$

Viscosity dominated by $\varphi + \varphi \rightarrow \varphi + \varphi$

$$\eta = A \frac{\xi^5}{c_s^3} \frac{T_F^8}{T^5}$$

**High T: Atoms** Cross section regularized by thermal momentum

$$\eta = \frac{15}{32\sqrt{2}} (mT)^{3/2}$$
$\eta/s$: Kinetic Theory

Low $T$ behavior very steep, no indication for smooth crossover.
Thermal conductivity

Superfluids are very efficient conductors of heat, by a process usually called superfluid convection.

There is a non-zero (but difficult to observe) diffusive contribution

\[ \vec{j}^c = -\kappa \vec{\nabla} T \]

The calculation of \( \kappa \) is subtle, because quasi-particles with linear dispersion \( E_p \sim c_sp \) do not contribute. [Roughly, linear qp’s always transport momentum together with energy.]

The dominant process is phonon splitting, made possible by non-linear terms in the dispersion relation.

\[
\kappa = \frac{128}{3\pi} \frac{\gamma^2}{g_3^2 c_s^2} T^2 D_H = \frac{256\sqrt{2}}{25\pi^3 \xi^2 m} (mT)^{3/2} \left( \frac{T}{T_F} \right)^2 D_H
\]

Normal phase \( \kappa \sim m^{1/2} T^{3/2} \)
Liquid Helium

Bosons, van der Waals + short range repulsion

\[ S = \int \Phi^\dagger \left( i\partial_t + \frac{\nabla^2}{2M} \right) \Phi + \int \int (\Phi^\dagger \Phi) V(x-y) (\Phi^\dagger \Phi) \]

with \( V(x) = V_{sr}(x) - c_6/x^6 \). Note: \( a = 189a_0 \gg a_0 \)

Phase Diagram

Excitations
**Low T: Phonons and Rotons** Effective lagrangian

\[
\mathcal{L} = \varphi^* (\partial_0^2 - v^2) \varphi + i \lambda \dot{\varphi} (\vec{\nabla} \varphi)^2 + \ldots \\
+ \varphi_{R,v}^* (i \partial_0 - \Delta) \varphi_{R,v} + c_0 (\varphi_{R,v}^* \varphi_{R,v})^2 + \ldots
\]

Shear viscosity

\[
\eta = \eta_R + \frac{c_{Rph}}{\sqrt{T}} e^{\Delta/T} + \frac{c_{ph}}{T^5} + \ldots
\]

**High T: Atoms** Viscosity governed by hard core \((V \sim 1/r^{12})\)

\[
\eta = \eta_0 (T/T_0)^{2/3}
\]
Experiment: Liquid Helium

Figure 1. The viscosity of liquid helium II measured by flow through a 10^{-4} cm channel.

Kapitza (1938)
viscosity vanishes below $T_c$
capillary flow viscometer

Hollis-Hallett (1955)
roton minimum, phonon rise
rotation viscometer
Experiments: Elliptic flow

Hydrodynamic expansion converts coordinate space anisotropy to momentum space anisotropy

O’Hara et al. (2002)
Determination of $\eta(n, T)$

Measurement of $A_R(t, E_0)$ determines $\eta(n, T)$. But:

The whole cloud is not a fluid.
Can we ignore this issue?

No. Hubble flow & low density viscosity $\eta \sim T^{3/2}$ lead to paradoxical fluid dynamics.

$$\dot{Q} = \int \sigma \cdot \delta \Pi = \infty$$

Not a fundamental problem (dilute corona is ballistic) but how to couple fluid and ballistic motion?
Possible approaches to dilute regime

1) Boltzmann equation for the entire cloud. Hard to incorporate $P(n, T)$ and $\eta(n, T)$.

2) Combine hydrodynamics & Boltzmann equation. Not straightforward.

3) Hydrodynamics + non-hydro degrees of freedom ($\mathcal{E}_a; a = x, y, z$)

$$\frac{\partial \mathcal{E}_a}{\partial t} + \vec{\nabla} \cdot \vec{j}^\epsilon_a = - \frac{\Delta P_a}{2\tau} \quad \Delta P_a = P_a - P$$

$$\frac{\partial \mathcal{E}}{\partial t} + \vec{\nabla} \cdot \vec{j}^\epsilon = 0 \quad \mathcal{E} = \sum_a \mathcal{E}_a$$

$\tau$ small: Fast relaxation to Navier-Stokes with $\tau = \eta/P$

$\tau$ large: Additional conservation laws. Ballistic expansion.
Anisotropic Hydrodynamics: Comparison with Boltzmann

Aspect ratio \( A_R(t) = (\langle r^2_\perp \rangle / \langle r^2_z \rangle)^{1/2} \) \( (T/T_F = 0.79, 1.11, 1.54) \)

Dots: Two-body Boltzmann equation with full collision kernel

Lines: Anisotropic hydro with \( \eta \) fixed by Chapman-Enskog

High temperature (dilute) limit: Perfect agreement!

Elliptic flow: High T limit

Quantum viscosity $\eta = \eta_0 \frac{(mT)^{3/2}}{\hbar^2}$

$T/T_F = 0.79, 1.11, 1.54$

Cao et al., Science (2010)
Bluhm et al., PRL (2016)

fit: $\eta_0 = 0.282 \pm 0.02$

theory: $\eta_0 = \frac{15}{32\sqrt{\pi}} = 0.269$
Anisotropic fluid dynamics analysis

$A_R = \sigma_x / \sigma_y$ as function of total energy. Data: Joseph et al (2016). $E/(NE_F) \sim 0.6$ is the superfluid transition.

Red, Blue, Green: LO, NLO, NNLO fit.

$$\eta = \eta_0 (mT)^{3/2} \left\{ 1 + \eta_2 n \lambda^3 + \eta_3 (n \lambda^3)^2 + \ldots \right\}$$

Converges rapidly above $T_c$. (Discontinuity below $T_c$?)
Reconstruct $\eta/n$ and $\eta/s$

Left: $\eta/n$ (Red band)

Right: $\eta/s$ (Red band) $T_c \sim 0.17 T_F$.

Joseph et al. (Black points). Enss et al. (Dashed line).

Kinetic theory at low and high T (blue dashed)

$$\eta(T \gg T_c) = (0.265 \pm 0.02)(mT)^{3/2}$$

$$\eta_0(th) = \frac{15}{32\sqrt{\pi}} = 0.269$$

$$\eta/n|_{T_c} = 0.41 \pm 0.15$$

$$\eta/s|_{T_c} = 0.56 \pm 0.20$$
Extend analysis to $T < T_c$


Analyzed using one-fluid dynamics. No sign of rise in $\eta/n$ as $T \to 0$. 
Outlook

Perform analysis using two-fluid hydrodynamics.

Data from homogeneous systems (box potentials).

Other transport properties: Spin diffusion, thermal conductivity, bulk viscosity near unitarity.