Anisotropic fluid dynamics

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We wish to extract the properties of nearly perfect (low viscosity) fluids from experiments with trapped gases, colliding nuclei, etc.

The natural tool for these studies is the Navier-Stokes equation, which describes the macroscopic motion of a fluid in which viscous corrections are small.

The problem is that this is not the case for the entire system. There is a dilute corona in which fluid dynamics is not applicable.
Hydrodynamics (undergraduate version): Newton’s law for continuous, deformable media.
Hydrodynamics (postmodern): Effective theory of non-equilibrium long-wavelength, low-frequency dynamics of any many-body system.

\[ \tau \sim \tau_{\text{micro}} \]

\[ \tau \gg \tau_{\text{micro}}: \text{Dynamics of conserved charges.} \]

Water: \((\rho, \varepsilon, \pi)\)
Consider a many body system (unitary Fermi gas) with $\sigma \sim 1/k^2$

Can be made using Feshbach resonances in dilute atomic gases.

Systems remains hydrodynamic despite expansion
Effective theories for fluids (Unitary Fermi Gas, $T > T_F$)

\[
\mathcal{L} = \psi^\dagger \left( i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2
\]

\[
\frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p = C[f_p] \quad \omega < T
\]

\[
\frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad \omega < T \frac{s}{\eta}
\]
Effective theories (Strong coupling)

\[ \mathcal{L} = \bar{\lambda}(i\sigma \cdot D)\lambda - \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \ldots \Leftrightarrow S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \mathcal{R} + \ldots \]

\[ SO(d + 2, 2) \rightarrow Schr^2_d \]

\[ AdS_{d+3} \rightarrow Schr^2_d \]

\[ \frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad (\omega < T) \]
Gradient expansion (simple non-relativistic fluid)

Simple fluid: Conservation laws for mass, energy, momentum

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j}^\rho = 0
\]

\[
\frac{\partial \epsilon}{\partial t} + \nabla \cdot \mathbf{j}^\varepsilon = 0
\]

\[
\frac{\partial \pi_i}{\partial t} + \frac{\partial}{\partial x_j} \Pi_{ij} = 0
\]

Ward identity: mass current = momentum density

\[
\mathbf{j}^\rho \equiv \rho \mathbf{v} = \mathbf{\pi}
\]

Constitutive relations: Gradient expansion for currents

Energy momentum tensor

\[
\Pi_{ij} = P \delta_{ij} + \rho v_i v_j + \eta \left( \partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right) + O(\partial^2)
\]
Gradient expansion, Kubo formula

Consider background metric \( g_{ij}(t, x) = \delta_{ij} + h_{ij}(t, x) \). Linear response

\[
\delta \Pi_{xy} = -\frac{1}{2} G_{R}^{xyxy} h_{xy}
\]

Harmonic perturbation \( h_{xy} = h_0 e^{-i\omega t} \)

\[
G_{R}^{xyxy} = P - i\eta\omega + \ldots
\]

Kubo relation: \( \eta = -\lim_{\omega \to 0} \left[ \frac{1}{\omega} \text{Im} G_{R}^{xyxy}(\omega, 0) \right] \)

Gradient expansion: \( \omega \leq \frac{P}{\eta} \sim \frac{s}{\eta} T \).
Fluid dynamics from kinetic theory

Microscopic picture:
Quasi-particle distribution function $f_p(x, t)$

![Diagram of particle distribution]

Boltzmann equation

$$
\left( \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}_x + \vec{F} \cdot \vec{\nabla}_p \right) f_p(t, x, ) = C[f_p]
$$

Collision term

$$
C[f_1] = \int d\Gamma_{234} (f_1 f_2 - f_3 f_4) w(12; 34)
$$
Fluid dynamics from kinetic theory

Conservation laws (collision term)

\[ \int d\Gamma_p \, M_p \, C[f_p] = 0 \quad M_p = \{1, p, E_p\} \]

Moments of Boltzmann equation imply fluid dynamic conservation laws

\[
\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j}^\rho = 0
\]
\[
\frac{\partial \varepsilon}{\partial t} + \vec{\nabla} \cdot \vec{j}^\varepsilon = 0
\]
\[
\frac{\partial \Pi_{ij}}{\partial t} + \frac{\partial}{\partial x_j} \Pi_{ij} = 0
\]

Need constitutive equations (and equation of state)

\[
\vec{j}^\rho = ? \quad \vec{j}^\varepsilon = ? \quad \Pi_{ij} = ?
\]
Kinetic theory: Knudsen expansion

Chapman-Enskog expansion \( f = f_0 + \delta f_1 + \delta f_2 + \ldots \)

\[
\text{Gradient exp. } \delta f_n = O(\nabla^n) \equiv \text{Knudsen exp. } \delta f_n = O(Kn^n)
\]

Zeroth order result: \( f_0 = \exp(-\beta(E_p - \vec{p} \cdot \vec{u} - \mu)) \) \( \beta = 1/T \)

\[
\tilde{j}^\rho = \tilde{\pi} = \rho\vec{u} \\
\tilde{j}^e = (\mathcal{E} + P)\vec{u} \quad P = \frac{2}{3}\mathcal{E} \\
\Pi_{ij} = \rho u_i u_j + P\delta_{ij}
\]

First order result: \( \delta f_1 = -f_0 \frac{\eta}{PT} v^i v^j \sigma_{ij} + \ldots \)

\[
\delta^{(1)}\Pi_{ij} = -\eta \sigma_{ij} \\
\delta^{(1)}j_i^e = -\eta u^j \sigma_{ij} - \kappa \nabla_i T
\]
Kinetic theory: Knudsen expansion

For given $w(12; 34)$ also obtain prediction for $\eta, \kappa$

$$\eta = \frac{15}{32 \sqrt{\pi}} (mT)^{3/2} \quad \kappa = \frac{225}{128 \sqrt{\pi m}} (mT)^{3/2}$$

Second order result

Chao, Schaefer (2012), Schaefer (2014)

$$\delta^{(2)} \Pi^{ij} = \frac{\eta^2}{P} \left[ \langle D\sigma^{ij} \rangle + \frac{2}{3} \sigma^{ij} (\nabla \cdot v) \right] + \frac{\eta^2}{P} \left[ \frac{15}{14} \sigma^{ik} \sigma_{jk}^k - \sigma^{ik} \Omega_{jk}^k \right] + O(\kappa \eta \nabla^i \nabla^j T)$$

relaxation time $\tau_\pi = \eta / P$
Experiments: Elliptic flow

Hydrodynamic expansion converts coordinate space anisotropy to momentum space anisotropy.

O’Hara et al. (2002)
Determination of $\eta(n, T)$

Measurement of $A_R(t, E_0)$ determines $\eta(n, T)$. But:

The whole cloud is not a fluid. Can we ignore this issue?

No. Hubble flow & low density viscosity $\eta \sim T^{3/2}$ lead to paradoxical fluid dynamics.

$$\dot{Q} = \int \sigma \cdot \delta \Pi = \infty$$
Possible Solutions

Combine hydrodynamics & Boltzmann equation. Not straightforward.

Hydrodynamics + non-hydro degrees of freedom \((\mathcal{E}_a; \ a = x, y, z)\)

\[
\frac{\partial \mathcal{E}_a}{\partial t} + \vec{\nabla} \cdot \vec{j}_a = -\frac{\Delta P_a}{2\tau} \quad \Delta P_a = P_a - P
\]

\[
\frac{\partial \mathcal{E}}{\partial t} + \vec{\nabla} \cdot \vec{j}^\epsilon = 0 \quad \mathcal{E} = \sum_a \mathcal{E}_a
\]

\(\tau\) small: Fast relaxation to Navier-Stokes with \(\tau = \eta/P\)

\(\tau\) large: Additional conservation laws. Ballistic expansion.
**Anisotropic hydro from kinetic theory**

Consider modified expansion

\[ f = f_A + \delta f_1' + \delta f_2' + \ldots \]

Anisotropic distribution function

\[ f_A = \exp \left( -\frac{(p_a - mu_a)^2}{2mT_a} - \frac{\mu}{\bar{T}} \right) \quad \bar{T} = (\prod T_a)^{1/3} \]

- \( f_A \) is an exact solution of the Boltzmann equation in the ballistic limit.
- The viscous stresses and dissipative corrections to the energy current have the same form as in the Chapman-Enskog theory.
Anisotropic Hydrodynamics from kinetic theory

Moments of the Boltzmann equation with $M_p = \{1, \vec{p}, E_P\}$.

Navier-Stokes with $\delta \Pi_{aa} = \Delta P_a$

Moments of the Boltzmann equation with $p_a^2$

$$\frac{\partial \mathcal{E}_a}{\partial t} + \vec{\nabla} \cdot \vec{j}_a = -\frac{\Delta P_a}{2\tau} \quad \Delta P_a = P_a - P$$

with $P_a = 2\mathcal{E}_a \ (P = \frac{2}{3} \mathcal{E})$ and $\tau = \eta/P$.

Solve fluid dynamic equations for small $\tau$

$$\delta \Pi_{aa} = \Delta P_a = -\eta \sigma_{aa}$$

Ballistic limit $\tau \rightarrow \infty$: Conservation law for $\mathcal{E}_a$. 
Consider $\eta = \alpha n$ and $\alpha \in [0, \infty)$

Navier-Stokes: Ideal hydro $\rightarrow$ very viscous hydro.

A-hydro: Ideal hydro $\rightarrow$ ballistic expansion.
Anisotropic Hydrodynamics: Evolution of $\delta \Pi_{aa}$

$$\eta = \alpha_n n$$

$$\eta = \alpha_T (mT)^{3/2}$$

$\delta \Pi_{xx}$ (Navier-Stokes)  $\delta \Pi_{xx}$ (A-Hydro)

AVH1 hydro code, M. Bluhm & T.S. (2015)
Anisotropic Hydrodynamics: Comparison with Boltzmann

\[ T/T_F = 0.79, 1.11, 1.54 \]

Dots: Two-body Boltzmann equation with full collision kernel

Lines: Anisotropic hydro with \( \eta \) fixed by Chapman-Enskog

High temperature (dilute) limit: Perfect agreement!

AVH1 hydro code, M. Bluhm & T.S. (2015)
Elliptic flow: High T limit

Quantum viscosity $\eta = \eta_0 \frac{(mT)^{3/2}}{\hbar^2}$

fit: $\eta_0 = 0.28 \pm 0.02$

theory: $\eta_0 = \frac{15}{32\sqrt{\pi}} = 0.269$

$T/T_F = 0.79, 1.11, 1.54$

Cao et al., Science (2010)
Bluhm et al., PRL (2016)
Outlook

Fluid dynamics as an E(F)T?

Unfold temperature, density dependence of $\eta/s$.

Applications to other transport problems: Diffusion, superfluid hydrodynamics.