Anomalies in Quantum Mechanics

GPSA Meeting – November 18, 2009

Outline

Symmetry breaking – explicit, spontaneous, or anomalous

Scale-invariance anomaly and the inverse square potential

Efimov effect in three-body bound states
Symmetric system

Dipoles
Symmetric system

Dipoles
Symmetric system

Dipoles
Explicit symmetry breaking

External field bias \(-\mu \vec{B} \cdot \sum_i \vec{s}_i\)
Spontaneous symmetry breaking

Spontaneous alignment

$$-J \sum_{<i,j>} \vec{s}_i \cdot \vec{s}_j$$
Spontaneous symmetry breaking

Spontaneous anti-alignment $+ J \sum_{<i,j>} \vec{s}_i \cdot \vec{s}_j$
Anomalous symmetry breaking

Simple analogy

Each person receives one dollar more than he/she spends

Total cash spent equals total cash received
Anomalous symmetry breaking

Massless fermions in one spatial dimension

\[ E = k_x \]

\[ E = -k_x \]
$N_{\to} = N_{\leftarrow}$

$k_x \to +\infty$

$\vdots$

$k_x \to -\infty$

$E$

$\vdots$

$k_x \to -\infty$

$\vdots$

$k_x \to +\infty$
Add constant vector potential

\[ k_x \rightarrow k_x - eA_x \]

\[ E = k_x - eA_x \]

\[ E = -(k_x - eA_x) \]
\[ \Delta E = -eA_x \]

\[ k_x \to +\infty \]
\[ \vdots \]

\[ k_x \to -\infty \]
\[ \vdots \]

\[ \Delta E = +eA_x \]

\[ k_x \to +\infty \]
\[ \vdots \]

\[ k_x \to -\infty \]
\[ N_{\to} \neq N_{\leftarrow} \]

\[ k_x \to +\infty \]
\[ \vdots \]
\[ k_x \to -\infty \]

\[ E \]
\[ 0 \]

\[ k_x \to -\infty \]
\[ \vdots \]
\[ k_x \to +\infty \]

\[ N_{\leftarrow} - N_{\to} \propto eA_x \]
Non-relativistic Quantum Mechanics
Separation of variables:

$$\Psi(r, \theta, \phi) = Y_{L,L_z}(\theta, \phi) R(r)$$

Radial Schrödinger equation:

$$-\frac{1}{2\mu r^2} \frac{d}{dr} \left[ r^2 \frac{d}{dr} R(r) \right] + \left[ \frac{L(L+1)}{2\mu r^2} + V(r) \right] R(r) = ER(r)$$

$$u(r) = rR(r)$$

$$-\frac{d^2}{dr^2} u(r) + \left[ \frac{L(L+1)}{r^2} + 2\mu V(r) \right] u(r) = 2\mu Eu(r)$$
$u(r)$

$V(r)$

$E$

$r$
Non-interacting particle

\[ V(r) = 0 \]

\[-\frac{d^2}{dr^2} u(r) + \frac{L(L+1)}{r^2} u(r) = 2\mu E u(r) \]

Positive energy:

\[ E > 0 \]

\[ E = \frac{p^2}{2\mu} \]

\[-\frac{d^2}{dr^2} u(r) + \frac{L(L+1)}{r^2} u(r) = p^2 u(r) \]

\[ u(r)\big|_{p} = \sqrt{pr} J_{L+1/2}(pr) \]
\[ \sqrt{pr} J_{\frac{1}{2}}(pr) \]

\[ L = 0: \]

\[ \sqrt{pr} J_{\frac{3}{2}}(pr) \]

\[ L = 1: \]
Invariance with respect to length scale

Scale invariance of the differential equation

\[ r \rightarrow r' = \lambda r, \quad p \rightarrow p' = p/\lambda \]

\[ -\frac{d^2}{dr^2} u + \frac{L(L+1)}{r^2} u = \frac{p^2}{\lambda^2} \]

Scale invariance of the solutions

\[ u(r)|_p = \sqrt{pr} J_{L+1/2}(pr) \]

\[ u(\lambda r)|_{p/\lambda} = u(r)|_p \]
Scale invariance of the spectrum

\[ p \rightarrow p' = p/\lambda \]

\[ E \rightarrow E' = E/\lambda^2 \]
Why not negative energy solutions?

\[ E < 0 \]

\[ E = -\frac{\kappa^2}{2\mu} \]

\[ -\frac{d^2}{dr^2} u(r) + \frac{L(L+1)}{r^2} u(r) = -\kappa^2 u(r) \]

Modified Bessel functions:

\[ u(r) \big|_{\kappa} = \sqrt{\kappa r} I_{L+1/2}(\kappa r)? \]

\[ u(r) \big|_{\kappa} = \sqrt{\kappa r} K_{L+1/2}(\kappa r)? \]

\[ I_{\alpha}(x) = i^{-\alpha} J_{\alpha}(ix) \]

\[ K_{\alpha}(x) = \frac{\pi}{2} i^{\alpha+1} \left[ J_{\alpha}(ix) + iY_{\alpha}(ix) \right] \]
\[ \sqrt{pr} \, I_{\frac{1}{2}}(pr) \]

diverges at infinity

\[ \sqrt{pr} \, K_{\frac{1}{2}}(pr) \]

cusp at zero

\[ \sqrt{pr} \, I_{\frac{3}{2}}(pr) \]

diverges at infinity

\[ \sqrt{pr} \, K_{\frac{3}{2}}(pr) \]

diverges at zero
Inverse square potential

\[ V(r) = \frac{c}{r^2} \]
Inverse square potential

\[- \frac{d^2}{dr^2} u(r) + \frac{L(L+1)}{r^2} u(r) + \frac{2\mu c}{r^2} u(r) = 2\mu E u(r)\]

\[- \frac{d^2}{dr^2} u(r) + \frac{\rho(\rho+1)}{r^2} u(r) = 2\mu E u(r)\]

\[\rho(\rho + 1) = L(L + 1) + 2\mu c\]

Scale invariance of the differential equation

\[r \rightarrow r' = \lambda r, \ E \rightarrow E' = E/\lambda^2\]

\[- \frac{d^2}{dr^2} u(r) + \frac{\rho(\rho+1)}{r^2} u(r) = 2\mu E u(r)\]

\[\times \frac{1}{\lambda^2} \times \frac{1}{\lambda^2} \times \frac{1}{\lambda^2}\]
Positive energy:

\[ E > 0 \]

\[ E = \frac{p^2}{2\mu} \]

\[ u(r) \bigg|_p = \sqrt{pr} J_{\rho+1/2}(pr) \]

Negative energy solutions?

\[ E < 0 \]

\[ E = -\frac{\kappa^2}{2\mu} \]

still diverges at infinity

\[ u(r) \bigg|_\kappa = \sqrt{\kappa r} I_{\rho+1/2}(\kappa r)? \]

\[ u(r) \bigg|_\kappa = \sqrt{\kappa r} K_{\rho+1/2}(\kappa r)? \]
Consider the attractive case, such that

\[ \rho(\rho + 1) = L(L + 1) + 2\mu c < -\frac{1}{4} \]

Define

\[ \alpha \equiv \rho + \frac{1}{2} \]

\[ \alpha^2 - \frac{1}{4} = \rho(\rho + 1) < -\frac{1}{4} \]

\[ \alpha^2 < 0 \]

The order of the modified Bessel function is pure imaginary

\[ u(r) \big|_{\kappa} = \sqrt{\kappa r} K_{\rho+1/2}(\kappa r) = \sqrt{\kappa r} K_{\alpha}(\kappa r) \]
Fractal behavior near zero

\[ K_i(pr) \]

\[ K_i(pr) \]

\[ K_i(pr) \]
Near zero

\[ K_\alpha(x) \propto \Re(x^{-\alpha}) \]

\[ x^{-\alpha} = e^{-\alpha \ln x} \]

For imaginary \( \alpha \)

\[ (xe^{i\pi \alpha})^{-\alpha} = e^{-\alpha \ln(xe^{i\pi})} \]

\[ = e^{-\alpha \ln x} e^{-i\pi} = -x^{-\alpha} \]

Discrete scale invariance

\[ r \to r' = \lambda r, \; E \to E' = E/\lambda^2 \]

\[ \lambda = e^{i\pi \alpha} \]
Bound state spectrum

\[ E \rightarrow E' = \frac{E}{\lambda^2} \]
\[ \lambda = e^{\frac{i\pi}{\alpha}} \]

Geometric series

\[ \cdots E_0\lambda^{-4}, E_0\lambda^{-2}, E_0, E_0\lambda^2, E_0\lambda^4, \cdots \]
The Efimov effect in three-body bound states


Consider zero-range two-body interactions tuned to zero-energy two-body resonance (infinite scattering length)
zero range, infinite scattering length
Cold atoms evaporatively cooled and trapped using lasers. Turn on an external magnetic field.

Zeeman tune energy of the diatomic molecule with external magnetic field to produce Feshbach resonance near threshold.
In the limit of zero-range two-body interactions with zero-energy two-body resonance

\[ R^2 = \vec{r}_{12}^2 + \vec{r}_{23}^2 + \vec{r}_{31}^2 \]

\[ V(R) \to -(4 + s_0^2) \frac{1}{2mR^2} \]

\[ s_0 \approx 1.00624 \]
\[ \lambda = e^{\frac{\pi}{s_0}} \approx 22.7 \]
\[ \lambda^2 \approx 515 \]

Geometric series of Efimov trimers

\[ \cdots E_0 \lambda^{-4}, E_0 \lambda^{-2}, E_0, E_0 \lambda^2, E_0 \lambda^4, \cdots \]

Confirmed in cold trap experiments using cesium atoms and also using potassium atoms


Summary

Anomalies are problems at infinity that can spoil or modify symmetries.

Anomalies are most often discussed in quantum field theory, but they also appear in simple quantum mechanics problems.