Instanton constituents in sigma models and Yang-Mills theory at finite temperature

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Extreme QCD, North Carolina State, July 2008

PRL 100 (2008) 051602 [0707.0775]

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Part I: The $O(3)$ sigma model

a scalar field in 2D ...

\[ S = \int d^2 x \frac{1}{2} (\partial_\mu \phi^a)^2 \quad a = 1, 2, 3 : \text{global } O(3) \text{ symmetry} \]

... with a constraint

\[ \phi^a \phi^a = 1 \quad (\text{circumvent Derrick's theorem}) \]

nontrivial properties:

- asymptotic freedom
- dynamical mass gap
- topology and instantons

condensed matter physics and toy model for gauge theories
finite action:

\[ r \to \infty : \quad \phi^a \to \text{const.} \]

as a mapping:

\[ \phi : \mathbb{R}^2 \cup \{\infty\} \simeq S^2_x \longrightarrow S^2_c \]

winding number/degree: all such \( \phi \)'s are characterized by an integer \( Q \)
= how often \( S^2_c \) is wrapped by \( S^2_x \) through \( \phi \)

here:

\[ Q = \frac{1}{8\pi} \int d^2 x \, \epsilon_{\mu\nu} \epsilon_{abc} \phi^a \partial_\mu \phi^b \partial_\nu \phi^c \in \mathbb{Z} \]

**topological quantum number** = invariant under small deformations of \( \phi \)
Classical solutions

Bogomolnyi trick...

\[
(\partial_\mu \phi^a \pm \epsilon_{\mu\nu} \epsilon_{abc} \phi^b \partial_\nu \phi^c)^2 = (\partial_\mu \phi^a)^2 \pm 2\epsilon_{\mu\nu} \epsilon_{abc} \phi^a \partial_\mu \phi^b \partial_\nu \phi^c + (\partial_\mu \phi^a)^2
\]

...and bound (integrated):

\[
S \geq 4\pi |Q|
\]

where the equality holds iff

\[
\partial_\mu \phi^a = \mp \epsilon_{\mu\nu} \epsilon_{abc} \phi^b \partial_\nu \phi^c \quad \text{‘selfduality equations’}
\]

first order (instead of second order in eqns. of motion)

classical solutions:

instantons = localised in both directions
Complex structure

introduce complex coordinates both in space and color space:

\[ x_{1,2} \to z = x_1 + ix_2 \]

\[ \phi^a \to u = \frac{\phi^1 + i\phi^2}{1 - \phi^3} \]

N: \( \phi^a = (0, 0, 1) \quad u = \infty \)

S: \( \phi^a = (0, 0, -1) \quad u = 0 \)

\[ \Rightarrow \text{self-duality equations become Cauchy-Riemann conditions on } u \]

\[ \Rightarrow \text{any meromorphic function } u(z) \text{ is a solution} \]

topological charge: \( Q = \text{number of zeroes or poles} \)

topological charge density:

\[ q(x) = \frac{1}{\pi} \frac{1}{(1 + |u|^2)^2} \left| \frac{\partial u}{\partial z} \right|^2 \]
Charge 1 instantons

- simplest functions:

\[
\begin{align*}
  u(z) &= \frac{\lambda}{z-z_0} \\
  u(z) &= \frac{z-z_0}{\lambda}
\end{align*}
\]

\[
q(x) = \frac{1}{\pi} \frac{\lambda^2}{(|z-z_0|^2+\lambda^2)^2}
\]

are \( Q = 1 \) instantons: location \( z_0 \), size \( \lambda \)

1 pole and 1 zero to cover \( S^2_c \), one of them at infinity

- both, pole and zero, at finite \( z \):

\[
u(z) = \frac{z - z_I}{z - z_{II}}
\]

constituents at \( z = \{z_I, z_{II}\} \) \( \sim \) ‘instanton quarks’?!

NO! same profile \( q(x) \) as above \( \Rightarrow \) one lump

conjecture: 2 complex moduli per \( Q \sim \) locations of 2 constituents?!
Finite temperature

= one compact direction, say: \( \text{Im } z = x_2 \sim x_2 + \beta, \quad \beta = 1/k_B T \)

• instantons:
  
  use that higher charge solutions = products

\[
u(z) = \prod_{k=1}^{Q} \frac{\lambda}{z - z_{0,k}} \quad \text{Q poles}
\]

and infinitely many copies: \( z_{0,k} \equiv z_0 + k \cdot i\beta, \quad k \in \mathbb{Z} \)

• a regularized \( u(z) \) is:

\[
u(z) = \frac{\lambda}{\exp((z - z_0) \frac{2\pi}{\beta}) - 1}
\]

has residues \( \lambda \) at \( z = z_0 + k \cdot i\beta \)

and charge 1 over \( S^1 \cdot R^1 \)

small \( \lambda \)

large \( \lambda \)
Boundary conditions

$q(x)$ and action density invariant under global $SO(3)$ rotations

an $SO(2)$ subgroup: $\phi \rightarrow \begin{pmatrix} \text{rotation} \\ \text{with } \omega \\ 1 \end{pmatrix} \phi, \quad u \rightarrow e^{2\pi i \omega} u$

• let the fields $\phi$ and $u$ be periodic up to that $SO(2)$ subgroup:

$$u(z + i \beta) = e^{2\pi i \omega} u(z) \quad \omega \in [0, 1]$$

$q(x)$ strictly periodic

• novel solution:

$$u(z) = \frac{e^{\omega(z-z_0) \frac{2\pi}{\beta}} \cdot \lambda}{\exp((z - z_0) \frac{2\pi}{\beta}) - 1}$$

has residues $e^{2\pi i \omega k} \lambda$ at $z = z_0 + k \cdot i \beta$

$\Rightarrow$ ‘different orientation’ of the instanton copies

$\Rightarrow$ nontrivial overlaps $\Rightarrow$ instanton constituents
Topological profiles

\( \ln q(x): \quad (z_0 = 0, \text{cut off below } e^{-5}) \)

\[ \lambda = \beta \quad \lambda = 10\beta \quad \lambda = 100\beta \]

periodic

\[ \omega = 1/3 \]

antiper.

\[ \Rightarrow \text{for large size } \lambda: \text{ 2 lumps with action } \omega \text{ and } \bar{\omega} = 1 - \omega \]
‘Dissociation’

• rewrite:

\[
    u(z) = \frac{1}{\exp(-\omega(z - z_1) \frac{2\pi}{\beta}) - \exp(\bar{\omega}(z - z_2) \frac{2\pi}{\beta})}
\]

locations: \( z_1 = z_0 - \beta \frac{\ln \lambda}{2\pi\omega} \), \( z_2 = z_0 + \beta \frac{\ln \lambda}{2\pi\bar{\omega}} \)

instanton size → constituent distance: \( z_2 - z_1 \sim \ln \lambda \)

constituent size: fixed by \( \beta \) and \( \omega \)

• really locations of topological lumps?

YES: corrections of the second term at \( z = z_1 \) are exp. small

• individual constituent:

\[
    u(z) = \exp(\omega z \frac{2\pi}{\beta}) \quad \bar{\omega} \text{ analogous}
\]

top. charge:

\[
    q(x) = \frac{\pi \omega^2}{\beta^2 \cosh^2(\omega \text{ Re } z \frac{2\pi}{\beta})} \quad \text{(static)} \quad Q = \omega
\]
• possible values for $Q$: $0, 1, \ldots \omega, 1 + \omega, \ldots, 1 - \omega, 2 - \omega, \ldots$

asympt. $\phi_{-\infty} \rightarrow \phi_{+\infty}$: $N \rightarrow N$, $S \rightarrow S$, $N \rightarrow S$, $S \rightarrow N$

constituents alternate

• why instanton quarks not visible for zero temperature, i.e. on $\mathbb{R}^2$?

$\beta \rightarrow \infty$: constituents large and overlap!

no other scale competing with their distance

• generalisations: $CP(N)$ models

FB et al. in progress

realisation in condensed matter:

cylinder of ..?.. with quasi-periodic bc.s
Fermionic zero modes

gauge field description $\Rightarrow$ couple fermions $\Rightarrow$ zero modes

phase-bc.s $\psi(x_0 + i\beta) = e^{2\pi i \zeta} \psi(x_0)$, evolution with $\zeta$: FB et al. in progress
pure Yang-Mills theory in (Euclidean) 4D:

\[ S = \int \frac{1}{2} \text{tr} \, F_{\mu \nu}^2 \geq |Q| = |\int \frac{1}{2} \text{tr} \, F_{\mu \nu} \tilde{F}_{\mu \nu}| \]

dual field strength
\[ \tilde{F}_{\mu \nu}^a = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} F^{a}_{\rho \sigma} \quad (\vec{E}^a \Leftrightarrow \vec{B}^a) \]

integer \( Q \): instanton number/topological charge

topology:
\[ A_{\mu} \xrightarrow{r \to \infty} i \Omega^{-1} \partial_\mu \Omega \quad \ldots \text{pure gauge} \]
\[ Q = \text{deg}(\Omega : S_r^3 \to SU(N)) \quad \ldots \text{winding number} \]
(anti)selfdual: $F^a_{\rho\sigma} = \pm \tilde{F}^a_{\mu\nu}$ first order, nonlinear

charge 1: axially symmetric ansatz and solution

$$A^a_\mu = \eta^a_{\mu\nu} \frac{2x_\nu}{x^2 + \rho^2} \quad \text{tr} F^2 = \frac{\rho^4}{(x^2 + \rho^2)^4} \quad \eta^a_{\mu\nu} \in \{-1, 0, 1\}$$

size $\rho$
localized in space and time
algebraic decay, similar to $O(3)$ instantons on $\mathbb{R}^2$

instanton liquid model from semiclassical path integral

- chiral symmetry breaking
- axial anomaly
- topological susceptibility
- confinement?
Finite temperature: Calorons

• use higher charge solutions of same color orientation
  \[ \Rightarrow \text{first calorons} \]  
  CFTW Harrington-Shepard ’78

• most general calorons: need ADHM formalism and Nahm transform
  \[ \Rightarrow \text{calorons of nontrivial holonomy} \]  
  Kraan, van Baal; Lee, Lu ’98

\[ \text{space-space plot of action density for } SU(2), \text{ intermediate holonomy} \]

\[ \Rightarrow 2 \text{ lumps, almost static} \]
\[ N_c \text{ for gauge group } SU(N_c), \text{ like quarks in baryons} \]

\[ \Rightarrow \text{magnetic monopoles of opposite magnetic charge} \]
\[ \text{in fact dyons with same electric as magn. charge (selfdual)} \]
Role of the holonomy

relative gauge orientation of instanton copies in the ADHM constr.

⇒ $A_\mu$ periodic up to a gauge transformation $e^{2\pi i \omega_3 / 2}$ (cf. $O(3)$)

gauge theory: compensated by time-dependent transf. $e^{2\pi i \omega_3 x_0 / 2}$

⇒ introduces an asymptotic gauge field $A_0$

⇒ asymptotic Polyakov loop = holonomy

$$\mathcal{P}(\vec{x}) \equiv \mathcal{P} \exp \left( i \int_0^\beta dx_0 A_0 \right) \to e^{2\pi i \omega_3 / 2} \equiv \mathcal{P}_\infty$$

‘environment’

acts like a Higgs field, in the group: vev $\omega$, direction $\sigma_3$

• monopoles have masses $\omega / \beta$ and $\bar{\omega} / \beta$, $\bar{\omega} = 1 - \omega$

• $A_{\mu}^{a=3}$: power law decay (massless ‘photon’),
  $A_{\mu}^{a=1,2}$: exponential decay (massive ‘$W$-bosons’)

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Instanton constituents

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• Polyakov loop in the bulk: $\mathcal{P}(\vec{x}) = \pm \frac{1}{2}$ at the monopoles
  Higgs field vanishes = ‘false vacuum’
  necessary for top. reasons
  Ford et al.; Reinhardt; Jahn et al.

• index theorem valid
  Nye, Singer
  localisation depending on bc.s:

  $$\psi(x_0 + i\beta) = e^{2\pi i \zeta} \psi(x_0) \quad (A_\mu \text{ still periodic})$$

  $\zeta \in \{-\frac{\omega}{2}, \frac{\omega}{2}\}$ incl. periodic: localised at monopole
  Garcia Perez et al.
  $\zeta \in \text{rest incl. antiperiodic: localised at antimonopole}$
  a zero in their profiles at the ‘other’ monopole, topological
  FB

• calorons can be studied on the lattice by cooling
  Ilgenfritz et al. ’02, FB et al.

• physical relevance of $\mathcal{P}_\infty$:
  conjecture: holonomy $\text{tr} \, \mathcal{P}_\infty \rightleftharpoons \text{deconfinement order param.} \langle \text{tr} \mathcal{P} \rangle_x$
Calorons and the dynamics of YM theories

- eff. potential at 1-loop: triv. holonomy favored! Gross, Pisarski, Jaffe; Weiss
  overruled by caloron gas contribution: Diakonov et al.
  ⇒ minima at $\mathcal{P} = \pm \frac{1}{2}$ become unstable for low enough temperature
  ⇒ onset of confinement

- gas of calorons and anticalorons put on the lattice: Gerhold et al.
  ⇒ linearly rising interquark potential just for nontrivial holonomy!

- confinement from a gas of purely selfdual dyons Diakonov, Petrov
  unphysical (top. charge builds up)
  nevertheless interesting physical effects
Summary

Sigma models in 2D and YM in 4D admit instantons

**Instanton constituents** for $S^1 \times R^1$ and $S^1 \times R^3 = \text{finite } T$

with fractional charges, say $\omega$ and $1 - \omega$ in the lowest models

when in compact direction **periodic up to a subgroup**, say $e^{2\pi i \omega}$.

= subgroup of **global** and **local** symmetry

Yang-Mills theory:

- can be made periodic $\rightarrow$ holonomy $\mathcal{P}_\infty$
- $\Rightarrow$ caloron constituents as building blocks of semiclass. models
  - at finite temperature: (de)confinement?!

Sigma models:

- quasi-periodic bc.s stay
- $\Rightarrow$ spin chains? skyrmion lattices? Quantum Hall effect?