The chiral critical surface of QCD for $\frac{\mu}{T} \lesssim 1$

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The situation at zero density

- \( N_f = 2, m = 0 \): \( N_t = 4 \), still not settled
  - ax. U(1) anomaly
  
  - phys. point: crossover in continuum
  
  - chiral critical line: two points on
    - \( N_t = 4 \)
    - \( N_t = 6 \)

\[ \begin{align*}
  N_f = 2, m = 0: & \quad N_t = 4, \text{ still not settled} \\
  \text{phys. point:} & \quad \text{crossover in continuum} \\
  \text{chiral critical line:} & \quad \text{two points on} \\
  \end{align*} \]

DiGiacomo et al. 05; Kogut, Sinclair 06; Chandrasekharan, Mehta 07

Aoki et al. 06

de Forcrand, O.P. 07

de Forcrand, O.P. 07; Endrodi et al. 07
Finite density: chiral critical line → critical surface

\[ \frac{m_c(\mu)}{m_c(0)} = 1 + \sum_{k=1}^{\infty} c_k \left( \frac{\mu}{\pi T} \right)^{2k} \]

\( c_1 > 0 \)

\( c_1 < 0 \)
How to identify the critical surface

\[ B_4(\bar{\psi}\psi) \equiv \frac{\langle (\delta \bar{\psi}\psi)^4 \rangle}{\langle (\delta \bar{\psi}\psi)^2 \rangle^2} \quad V \to \infty \]

\[ \begin{cases} 
1.604 & 3 \text{d Ising} \\
1 & \text{first-order} \\
3 & \text{crossover}
\end{cases} \]

\[ \mu = 0 : \]

\[ B_4(m, L) = 1.604 + b L^{1/\nu}(m - m_c^0), \quad \nu = 0.63 \]

\[ \mu = 0 : \]

\[ B_4(m, L) = 1.604 + b L^{1/\nu}(m - m_c^0), \quad \nu = 0.63 \]
Observable expanded about chiral critical point

\[ B_4(am, a\mu) = 1.604 + \sum_{k,l=1} b_{kl} (am - am_0^c)^k (a\mu)^{2l} \]

\[
c'_1 = \frac{d am_c}{d(a\mu)^2} = -\frac{\partial B_4}{\partial(a\mu)^2} \left( \frac{\partial B_4}{\partial am} \right)^{-1} = -\frac{b_{01}}{b_{10}},
\]

\[
c'_2 = \frac{d^2 am_c}{d[(a\mu)^2]^2} = \ldots = -\frac{b_{02}}{b_{10}} + \frac{b_{01} b_{11}}{b_{10}^2},
\]

continuum conversion: (requires beta-function)

\[
c_1 = \frac{\pi^2}{N_t^2} \frac{c'_1}{am_0^c} + \frac{1}{T_c(m_0^c, 0)} \frac{dT_c(m^c(\mu), \mu)}{d(\mu/\pi T)^2},
\]

\[
c_2 = c_1^2 + \left( \frac{\pi}{N_t} \right)^4 \left( \frac{c'_2}{am_0^c} - \frac{c'_1^2}{(am_0^c)^2} \right) - \frac{1}{T_c^2(m_0^c, 0)} \left( \frac{dT_c(m^c(\mu), \mu)}{d(\mu/\pi T)^2} \right)^2
\]

\[+ \frac{1}{T_c^2(m_0^c, 0)} \frac{d^2 T_c(m^c(\mu), \mu)}{d[(\mu/\pi T)^2]^2}.\]
Two methods to extract Taylor coefficients

I. Calculate at imaginary chem. potential, fit to truncated polynomial

\[ \langle O \rangle = \sum_{n}^N c_n \left( \frac{\mu_i}{\pi T} \right)^{2n} \Rightarrow \mu_i \rightarrow i\mu_i \]

de Forcrand, O.P., JHEP 07: \( N_t = 4 \) \( c_1 < 0 \) exotic scenario!

contradicts expectations, Fodor,Katz 04

Truncation errors, potentially dangerous:

\[ O = o_0 - o_1 \left( \frac{\mu_i}{\pi T} \right)^2 + o_2 \left( \frac{\mu_i}{\pi T} \right)^4 - o_3 \left( \frac{\mu_i}{\pi T} \right)^6 + \ldots \]

\[ O = o_0 + o_1 \left( \frac{\mu_r}{\pi T} \right)^2 + o_2 \left( \frac{\mu_r}{\pi T} \right)^4 + o_3 \left( \frac{\mu_r}{\pi T} \right)^6 + \ldots \]

Finite order fits “average” over higher terms
II. Calculate Taylor coefficients directly

Bielefeld-Swansea; Gavai, Gupta; MILC:

express derivatives by traces of non-local operators, \( f(\det M) \),
evaluate by stochastic estimators \[ \text{delicate cancellations!} \]

de Forcrand, Kim, O.P. (LAT07):

\[
\frac{dO}{d(a\mu)^2} = \lim_{(a\mu)^2 \to 0} \frac{O(a\mu) - O(0)}{(a\mu)^2}
\]

evaluate by “infinitesimal” reweighting, no overlap problem,
correlated errors drop out of observables

reweight in imaginary direction: reweighting factor real

compute reweighting factor by stochastic estimator

numerically very efficient
Numerical results for $N_f = 3$, $N_t = 4$

unimproved staggered fermions, RHMC algorithm

Method I: $8^3 \times 4, 42$ pairs $(am, a\mu) > 20$ million traj., 18 unconstrained dof's in fits

Method II: $8^3, 12^3 \times 4$ $m_{\pi}L \gtrsim 3, 4.5 > 5$ million, 0.5 million traj.

Mutually consistent; significant NLO-contribution!
Finite size scaling

scaling: each term $\propto L^{1/\nu}; \nu = 0.63$

LO and NLO close to thermodynamic limit
Combination of both methods

complementary techniques to extract coefficients → combine!

\[
\frac{B_4(\mu_i) - B_4(0)}{\mu_i^2} = b_1 + b_2 \mu_i^2 + b_3 \mu_i^4 + b_4 \mu_i^6
\]

- Derivative data
- Derivative data
- NLO derivative
- NNLO joint
- NNNLO joint
Converting to the continuum

- fits to imag. \( \mu \) alone
  \[
  \frac{m_c(\mu)}{m_c(0)} = 1 - 2.1(7) \left( \frac{\mu}{\pi T} \right)^2 - 9(5) \left( \frac{\mu}{\pi T} \right)^4 + \ldots
  \]

- derivatives alone
  \[
  \frac{m_c(\mu)}{m_c(0)} = 1 - 3.3(3) \left( \frac{\mu}{\pi T} \right)^2 - 43(22) \left( \frac{\mu}{\pi T} \right)^4 + \ldots
  \]

- combined, \( b_3=0 \)
  \[
  \frac{m_c(\mu)}{m_c(0)} = 1 - 3.3(3) \left( \frac{\mu}{\pi T} \right)^2 - 20(5) \left( \frac{\mu}{\pi T} \right)^4 + \ldots
  \]

- combined, \( b_3 \) released
  \[
  \frac{m_c(\mu)}{m_c(0)} = 1 - 3.3(3) \left( \frac{\mu}{\pi T} \right)^2 - 43(20) \left( \frac{\mu}{\pi T} \right)^4 - \ldots
  \]

Higher order corrections reinforce shrinking of first order region!
Non-degenerate fermion masses, \( N_f = 2 + 1, N_t = 4 \)

\[ \Delta B4/\Delta \mu_i^2 \]

16\(^3\) × 4, \( a m_s = 0.25, a m_{u,d} = 0.005, m_\pi L \sim 3 \) lighter than in nature 350k traj.

\[ b_1 = -66(41) (\mu^2 \text{ fit}), \quad b_1 = -71(75) (\mu^2 + \mu^4 \text{ fit}) \]

\[ \frac{m_c(\mu)}{m_c(0)} = 1 - 80(50) \left( \frac{\mu}{\pi T} \right)^2 - \ldots \] not conclusive yet
Towards the continuum limit $N_f = 3, N_t = 6, \mu = 0$

\[ \frac{m_c^c(N_t = 4)}{m_c^c(N_t = 6)} \approx 1.77 \approx \sqrt{3} \]
Towards the continuum limit \( N_f = 3, N_t = 6, \mu \neq 0 \)

\[
\frac{\partial B_4}{\partial am} \quad \text{easy, from fits to m-dependence} \quad \frac{\partial B_4}{\partial (a\mu)^2} \quad \text{hard, finite differences}
\]

\[
\Delta B_4/\Delta \mu^2
\]

\[
b_1 = -58(49) (\mu^2 \text{ fit}), \quad b_1 = -88(75) (\mu^2 + \mu^4 \text{ fit})
\]

\[
\frac{m_c(\mu)}{m_c(0)} = 1 - 28(23) \left( \frac{\mu}{\pi T} \right)^2 - \ldots
\]

Assume \( c_1 = +18 \) (+two sigma) \( \frac{m_c(\mu = T)}{m_c(0)} \approx 3 \) still no critical point!
Conclusions

- $N_t = 4$: exotic scenario without chiral critical point established for \( N_f = 3 \)
- reinforced by subleading terms in \( \left( \frac{\mu}{T} \right)^2 \)
- so far no qualitative change for \( N_f = 2 + 1 \)
- $N_t = 6$: sign undetermined so far, but curvature of crit. surface too small for critical point at \( \mu \lesssim T_c \)

**Caveat:** all staggered, masses lighter than physical, rooting problems?
LQCD on the Computing Grid

- 725k trajectories (2 quark masses) in 2 months → 115 CPU years
- on average 700 CPUs active at all times
- 330k files = 3 TB of data transferred
- computing support provided by CERN IT/GS: thanks a lot!

- calculations on EGEE Grid
- resources provided by CERN, CYFRONET (Poland), CSCS (Switzerland), NIKHEF (Holland) + 10 more across Europe
Contradiction with other lattice studies? ...not necessarily!

- Gavai & Gupta: $N_f = 2$ ‘miles away’
- Fodor & Katz: $\{T_E, \mu_E\} = \{162(2), 120(13)\}$ MeV

not same parameters, different systematics, lattice spacing effects

F&K keep $(am_q)$ fixed, but $a(\mu)$ increases with $\mu$

$\Rightarrow$ unphysically light quarks at larger $\mu$ may cause the phase transition