Chiral fermions and chemical potential
Some thoughts

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Introduction of chemical potential on the lattice for chiral fermions

- Overlap Dirac operator at nonzero chemical potential and random matrix theory; Jacques Bloch, Tilo Wettig; hep-lat/0604020

- Domain-wall and overlap fermions at nonzero quark chemical potential; Jacques Bloch, Tilo Wettig; arXiv:0709.4630 [hep-lat]

- Energy density for chiral lattice fermions with chemical potential; Christof Gattringer, Ludovit Liptak; arXiv:0704.0092 [hep-lat]

- Thermodynamics of the ideal overlap quarks on the lattice; Debasish Banerjee, R.V. Gavai, Sayantan Sharma; arXiv:0803.3925 [hep-lat]

**Basic idea in hep-lat/0604020**

  Replace $U_4$ by $e^{i\mu}U_4$ and $U_4^\dagger$ by $e^{-i\mu}U_4^\dagger$ everywhere in the Wilson-Dirac kernel.

- Extend the definition of the sign function.
Massless overlap Dirac operator

Massless overlap Dirac operator in even dimensions is

$$D_o = \frac{1}{2} (1 + \gamma_{d+1} \epsilon[H_w(U, m)])$$

- $H_W$ is the hermitian Wilson Dirac operator.
- $U$ is the background gauge field.
- $m$ is the negative Wilson mass taken to be in the range $[0, 2]$.
- $\epsilon$ function on a Hermitian matrix is defined as follows: If
  $$H_w = V \Lambda V^\dagger$$
  where $V$ is the unitary matrix that diagonalizes $H_w$ and $\Lambda_{ij} = \lambda_i \delta_{ij}$ with real $\lambda_i$ being the eigenvalues of $H_w$; then
  $$\epsilon(H_w) = V \frac{\Lambda}{|\Lambda|} V^\dagger$$
  where $|\Lambda|_{ij} = |\lambda_i| \delta_{ij}$. 
Wilson Dirac operator with a chemical potential

In the presence of a chemical potential, \( \mu \), \( H_w \) is not hermitian. It takes the form

\[
H_w = \begin{pmatrix} B & C_R \\ C_L & -B \end{pmatrix}
\]

\[
[C_L]_{x\alpha i, y\beta j} = \frac{1}{2} \sum_{k=1}^{d-1} \sigma_5^{\alpha \beta} \left[ \delta_{y, x+k}(U_k(x))_{ij} - \delta_{x, y+k}(U_k^\dagger(y))_{ij} \right] + \frac{1}{2} \sigma_d^{\alpha \beta} \left[ \delta_{y, x+\hat{d}}e^\mu(U_d(x))_{ij} - \delta_{x, y+\hat{d}}e^{-\mu}(U_d^\dagger(y))_{ij} \right]
\]

\[
[C_R]_{x\alpha i, y\beta j} = -\frac{1}{2} \sum_{k=1}^{d-1} \sigma_5^{\dagger \alpha \beta} \left[ \delta_{y, x+k}(U_k^\dagger(x))_{ij} - \delta_{x, y+k}(U_k^\dagger(y))_{ij} \right] - \frac{1}{2} \sigma_d^{\dagger \alpha \beta} \left[ \delta_{y, x+\hat{d}}e^\mu(U_d(x))_{ij} - \delta_{x, y+\hat{d}}e^{-\mu}(U_d^\dagger(y))_{ij} \right]
\]

\[
[B]_{x\alpha i, y\beta j} = \frac{1}{2} \delta_{\alpha \beta} \sum_{k=1}^{d-1} \left[ 2\delta_{xy}\delta_{ij} - \delta_{y, x+k}(U_k(x))_{ij} - \delta_{x, y+k}(U_k^\dagger(y))_{ij} \right] + \frac{1}{2} \delta_{\alpha \beta} \left[ 2\delta_{xy}\delta_{ij} - \delta_{y, x+\hat{d}}e^\mu(U_d(x))_{ij} - \delta_{x, y+\hat{d}}e^{-\mu}(U_d^\dagger(y))_{ij} \right] - m\delta_{x\alpha i, y\beta j}
\]

- \( C_L^{\dagger}(\mu) = C_R(-\mu) \)
- \( B^{\dagger}(\mu) = B(-\mu) \)
The definition of the $\epsilon$ function for a hermitian matrix is extended to a general complex matrix as follows: If

$$H_w = V \Lambda V^{-1}$$

where $V$ is a complex matrix that diagonalizes $H_w$ and $\Lambda_{ij} = \lambda_i \delta_{ij}$ with complex $\lambda_i$ being the eigenvalues of $H_w$; then

$$\epsilon(H_w) = V \epsilon(\Lambda) V^{-1} = V \left[ \lim_{L_s \to \infty} \frac{e^{L_s \Lambda} - 1}{e^{L_s \Lambda} + 1} \right] V^{-1}$$

where $\left[ e^{L_s \Lambda} \right]_{ij} = e^{L_s \lambda_j} \delta_{ij}$.

If

$$\lambda_j = R_j + i I_j$$

then

$$\lim_{L_s \to \infty} \frac{e^{L_s \lambda_j} - 1}{e^{L_s \lambda_j} + 1} V^{-1} = \frac{R_j}{|R_j|} = \frac{\text{Re} \lambda_j}{|\text{Re} \lambda_j|}$$
A domain-wall justification

The domain wall action for massless fermions can be written as (H. Neuberger, hep-lat/9710089)

\[
S = - \sum_{s=1}^{2L_s} \bar{\Phi}_s (D \Phi)_s
\]

\[
\begin{pmatrix}
\bar{\Phi}_1 & \bar{\Phi}_2 & \cdots & \bar{\Phi}_{2L_s-1} & \bar{\Phi}_{2L_s}
\end{pmatrix}
= \begin{pmatrix}
\bar{\chi}^R_1 & \bar{\chi}^L_1 & \cdots & \bar{\chi}^R_{L_s-1} & \bar{\chi}^L_{L_s}
\end{pmatrix}
\]

\[
D = \begin{pmatrix}
C_R & B + 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
B + 1 & -C_L & -1 & 0 & 0 & \cdots & 0 & 0 \\
0 & -1 & C_R & B + 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & B + 1 & -C_L & -1 & \cdots & 0 & 0 \\
0 & 0 & 0 & -1 & C_R & B + 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & B + 1 & -C_L & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 0 & \cdots & B + 1 & -C_L \\
\end{pmatrix}
\]

The physical fermion is

\[
\bar{\psi} = \begin{pmatrix}
\bar{\chi}^R_1 & \bar{\chi}^L_{L_s}
\end{pmatrix}
\]
Pseudofermions

The contribution from all the unphysical fermions are subtracted by the pseudofermion action

\[ D_{pf} = \begin{pmatrix}
C_R & B + 1 & 0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 1 \\
B + 1 & -C_L & -1 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\
0 & -1 & C_R & B + 1 & 0 & 0 & \ldots & \ldots & 0 & 0 \\
0 & 0 & B + 1 & -C_L & -1 & 0 & \ldots & \ldots & 0 & 0 \\
0 & 0 & 0 & -1 & C_R & B + 1 & \ldots & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & B + 1 & -C_L & \ldots & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots & B + 1 & -C_L \\
\end{pmatrix} \]
Fermion determinant

\textit{Neuberger, hep-lat/9710089}

\[
\det D = (\det B)^k \det \left[ \frac{1 - T^{-k}}{2} - \frac{1 + T^{-k}}{2} \gamma_5 \right]
\]

\[
\det D_{\text{pf}} = (\det B)^k \det \left[ - \left( 1 + T^{-k} \right) \gamma_5 \right]
\]

\[
T = \begin{pmatrix}
\frac{1}{B+1} & \frac{1}{B+1} C_L \\
C_R \frac{1}{B+1} & C_R \frac{1}{B+1} C_L + B + 1
\end{pmatrix}
\]

\[
\frac{\det D}{\det D_{\text{pf}}} = \det \frac{1}{2} \left[ 1 + \gamma_5 \tanh \frac{T^{-L_s} - 1}{T^{-L_s} + 1} \right]
\]
Using \textit{tanh} to define $\epsilon$

$T$ is a general complex matrix in the presence of a chemical potential and we are interested in

\[
\frac{T^{-L_s} - 1}{T^{-L_s} + 1}
\]

Let

\[
T = VEV^{-1}
\]

where $V$ is the general complex matrix that diagonalizes $T$ and $E_{ij} = e_i \delta_{ij}$ is the diagonal matrix made up of the complex eigenvalues, $e_i$, of $T$.

Then

\[
\frac{T^{-L_s} - 1}{T^{-L_s} + 1} = V \frac{E^{-L_s} - 1}{E^{-L_s} + 1} V^{-1}
\]

\[
\lim_{L_s \to \infty} \frac{e_i^{-L_s} - 1}{e_i^{-L_s} + 1} = \begin{cases} 
1 & \text{if } |e_i| < 1 \\
-1 & \text{if } |e_i| > 1
\end{cases}
\]
Overlap Dirac operator with chemical potential

\[\lim_{L_s \to \infty} \frac{T^{-L_s} - 1}{T^{-L_s} + 1} = \epsilon [- \ln T]\]

\[\lim_{L_s \to \infty} \frac{\det D}{\det D_{pf}} = \det D_o\]

\[D_o = \frac{1}{2} [1 + \gamma_5 \epsilon (- \ln T)]\]

\[- \ln T \to H_w\] as the lattice spacing in the \((d + 1)\) direction goes to zero.

The definition of the overlap Dirac operator with a chemical potential in hep-lat/0604020 seems justified.
Isospin chemical potential

\[ H_w(\mu) = \begin{pmatrix} B(\mu) & C_R(\mu) \\ C_L(\mu) & -B(\mu) \end{pmatrix} \]

- \[ C_L^\dagger(\mu) = C_R(-\mu) \]
- \[ B^\dagger(\mu) = B(-\mu) \]

\[ H_w^\dagger(\mu) = H_w(-\mu) \]

\[ \epsilon \left( H_w(-\mu) \right) = [\epsilon \left( H_w(\mu) \right)]^\dagger \]

\[ \text{det } D_o(\mu) = [\text{det } D_o(-\mu)]^* \]
A reminder of the derivation of the overlap Dirac operator

Overlap fermions provides a solution to the problem of putting chiral fermions on the lattice.

Assume there is no chemical potential.

Form two many body operators:

\[ \mathcal{H}_- = a^\dagger \gamma_5 a \]
\[ \mathcal{H}_+ = a^\dagger H_w a \]

Then

\[ \det C_L = \langle b - | b+ \rangle \]
\[ \det C_R = \langle t - | t+ \rangle \]

where \( | b\pm \rangle \) are the normalized lowest energy states of \( \mathcal{H}_\pm \) and \( | t\pm \rangle \) are the normalized highest energy states of \( \mathcal{H}_\pm \).

Phases of these states have to be fixed such that

\[ \det C_L = \det C_R^{\dagger} \]
The computation

Let

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be the unitary matrix that diagonalizes $H_w$ with $\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}$ and $\begin{pmatrix} \beta \\ \delta \end{pmatrix}$ spanning the positive and negative eigenvalues of $H_w$ respectively.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ span the positive and negative eigenvalues of $\gamma_5$ respectively.

Therefore, $\det C_L = \delta$ and $\det C_R = \alpha$ up to a phase.

Since $V$ is unitary, one can show that

$$\det V = \frac{\det \alpha}{\det \delta^\dagger}$$

Since $\det V \det V^\dagger = 1$, it follows that

$$\det \alpha \det \alpha^\dagger = \det \delta \det \delta^\dagger$$

and therefore $\det C_L \det C_L^\dagger = \det C_R \det C_R^\dagger$ are the same and independent of the phase choice.
Derivation of the overlap Dirac operator

\[ \epsilon(H_w)V = \begin{pmatrix} \alpha & -\beta \\ \gamma & -\delta \end{pmatrix} \]

\[ \gamma_5\epsilon(H_w)V = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix} \]

\[ D_oV = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \]

\[ \det D_o \frac{\det \alpha}{\det \delta^\dagger} = \det \alpha \det \delta \]

\[ \det D_o = \det \delta \det \delta^\dagger \]
Addition of the chemical potential

\[ V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \]

\[ \epsilon(H_w)V = \begin{pmatrix} \alpha & -\beta \\ \gamma & -\delta \end{pmatrix} \]

\[ \gamma_5\epsilon(H_w)V = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix} \]

\[ D_o V = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \]

\[ \det D_o \det V = \det \alpha \det \delta \]

\[ \det D_o = \det V^{-1} \det \alpha \det \delta \]
Remarks

- $H_w$ is not hermitian. $\mathcal{H}_+$ is not a hermitian many body operator.

- $a^\dagger$ should really be replaced by $a^{-1}$. It carries the same meaning. $a^{-1}$ is the creation operator and is the inverse of $a$, the annihilation operator.

- If $(a, a^{-1})$ obey canonical anticommutation relations, and if $b = V^{-1}a$, then $(b, b^{-1})$ also obey canonical anticommutation relations.

- $\det C_L \det C_R$ is not real and positive.

- There should be no ambiguity in the definition of $\det C_L \det C_R$.

- Under $V \rightarrow DV$ where $D$ is an arbitrary complex diagonal matrix, $\det V^{-1} \det \alpha \det \delta$ and therefore $\det D_o$ is invariant.

- The propagator

  $$G_o = D_o^{-1} - 1 = \begin{pmatrix} 0 & \beta \delta^{-1} \\ \gamma \alpha^{-1} & 0 \end{pmatrix}$$

  is clearly chiral and is invariant under $V \rightarrow DV$. 
Eigenvalues of $S = \gamma_5 \epsilon$

$$\epsilon^2 = 1$$

Let

$$S\psi = s\psi$$

Then

$$\epsilon\psi = s\gamma_5\psi \Rightarrow \psi = s\epsilon\gamma_5\psi \Rightarrow \frac{1}{s}[\gamma_5\psi] = S[\gamma_5\psi]$$

- There is a pairing of eigenvalues of the form, $(s, 1/s)$.
- $s = \pm 1$ are not paired.
- $s = -1$ corresponds to a zero mode of $D_o$.
- If $\epsilon$ is hermitian, $S$ is unitary and all eigenvalues lie on the unit circle.
- In the presence of $\mu$, eigenvalues inside the unit circle have partners outside the unit circle.
\begin{align*}
\det D_o(\mu) \\
\text{Assume we are in the zero topological sector.} \\
\text{When } \mu = 0, \text{ let } s_j = e^{i\phi_j} \text{ with } 0 \leq \phi_j < \pi \text{ be half the eigenvalues of } S. \text{ Then,} \\
\det D_o(0) = \prod_j \cos^2 \frac{\phi_j}{2} \\
\text{When } \mu \neq 0, \text{ let us assume that } |s_j| < 1 \text{ be half the eigenvalues of } S. \text{ Then,} \\
\det D_o(0) = \prod_j \frac{1}{4} \left[ 2 + s_j + s_j^{-1} \right] \\
\text{Whether } \mu = 0 \text{ or } \mu \neq 0, s_j \text{ close to } -1 \text{ cause a suppression and this is just the role of almost zero modes.} \\
\text{What if all the } s_j \text{ with } |s_j| < 1 \text{ get close to zero? (A possible scenario as } \mu \text{ is increased)} \\
\text{Then, the determinant gets very large (opposite of suppression).} \\
\text{In addition, if the phase of } s_j \text{ gets uniformly distributed on the unit circle, then the determinant will remain large.}
\end{align*}
Phase of $\det D_{O}(\mu)$

- Phase of the fermion determinant results in the sign problem.

- What happens in large $N_c$ QCD with finite number of fermions flavors?

- Can we work in the quenched approximation even with a chemical potential in the large $N_c$ limit?

- The fermion determinant should still be one power of $N_c$ less than the vacuum polarization.

- But the fermion determinant will have a factor of $N_c$ and this implies that the phase of the determinant can be anywhere on the unit circle. If so, phase averaging is a problem.

- What is the phase distribution of the eigenvalues $s_j$ with $|s_j| > 1$?