Revisiting the strong coupling limit of lattice QCD

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Motivation

- 25\textsuperscript{+} years of analytic predictions:
  
  80's: Kluberg-Stern et al., Kawamoto-Smit, Damgaard-Kawamoto
  
  \[ T_c(\mu = 0) = \frac{5}{3}, \quad \mu_c(T = 0) = 0.66 \]

  90's: Petersson et al., 1/g\textsuperscript{2} corrections

  00's: detailed (\(\mu, T\)) phase diagram: Nishida, Kawamoto,...

  08: Ohnishi, Münster & Philipsen,...

  How accurate is mean-field (1/d) approximation?

- Almost no Monte Carlo crosschecks:
  
  89: Karsch-Mütter \(\rightarrow\) MDP formalism \(\rightarrow\) \(\mu_c(T = 0) \sim 0.63\)

  92: Karsch et al. \(T_c(\mu = 0) \approx 1.40\)

  99: Azcoiti et al., MDP ergodicity ??

  06: PdF-Kim, HMC \(\rightarrow\) hadron spectrum \(\sim 2\%\) of mean-field

Can one trust the details of analytic phase-diagram predictions?
Phase diagram according to Nishida (2004)

Very similar to conjectured phase diagram of $N_f = 2$ QCD
Strong coupling $SU(3)$ with staggered quarks

$$Z = \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp(-\bar{\psi}(\not{D}(U) + m)\psi), \text{ no plaquette term (} \beta = 0 \text{)}$$

- One KS fermion field (ie. 4 “tastes”): 6 d.o.f. per site
- $\not{D}(U) = \frac{1}{2} \sum_{x,\nu} \eta_\nu(x)(U_\nu(x) - U_\nu^\dagger(x - \hat{\nu}))$, $\eta_\nu(x) = (-)^{x_1 + \ldots + x_{\nu-1}}$
- Chemical potential $\mu \rightarrow \exp(\pm a\mu) U_{\pm4}$
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- **Alternative 1:** integrate over fermions
  $$Z = \int \mathcal{D} U \det(\bar{\mathcal{D}}(U) + m) \rightarrow \text{HMC, etc...}$$
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  $$Z = \int \mathcal{D} U \det(\bar{D}(U) + m) \rightarrow \text{HMC, etc...}$$

- Alternative 2: integrate over links
  $$\rightarrow \text{Color singlet degrees of freedom:}$$
  - Monomer (meson $\bar{\psi}\psi$) $M(x) \in \{0, 1, 2, 3\}$
  - Dimer (meson hopping), non-oriented $n_{\nu}(x) \in \{0, 1, 2, 3\}$
  - Baryon hopping, oriented $\bar{B}B_{\nu}(x) \in \{0, 1\} \rightarrow \text{self-avoiding loops } C$
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$$Z(m,\mu) = \sum_{\{M,n_\nu,C\}} \prod_x \frac{3!}{M(x)!} m^{M(x)} \prod_{x,\nu} \frac{(3 - n_\nu(x))!}{3!n_\nu(x)!} \prod_{\text{loops } C} \rho(C)$$

with constraint $(M + \sum_{\nu} n_\nu)(x) = 3 \ \forall x \notin \{C\}$
MDP Monte Carlo

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with constraint \((M + \sum_{\pm v} n_v)(x) = 3 \forall x \notin \{C\}\]

3 difficulties:

- **sign** of \(\prod_C \rho(C)\):
  - associate \(\pm\) baryon loops with (1212.. & 2121..) polymer loops
  - weight: \(\pm \cosh \frac{\mu}{T} + 1 \rightarrow\) much milder sign problem

**MDP ensemble**

\[\text{Karsch & Mütter}\]
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  - MDP ensemble
    - Karsch & Mütter

- **changing monomer number difficult:** weight $\sim m^{\sum_x M(x)}$
  - monomer-changing update (Karsch & Mütter) restricted to $m \sim O(1)$
MDP Monte Carlo

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- **tight-packing constraint** → local update inefficient, esp. as \(m \to 0\)
**MDP Monte Carlo**

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\]

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Solved with **worm algorithm** (Prokof’eev & Svistunov)
Worm algorithm for MDP

Here for chiral limit $m = 0$ (no monomers: $M(x) = 0 \ \forall x$)

- Break a dimer bond and introduce a pair of adjacent monomers $M(x), M(y)$
- Choose among neighbours of $y$ by local heatbath and move $M(y)$ there
  
  heatbath: sampling of 2-point function $\frac{1}{Z} M(x)M(y)\exp(-S)$
- Keep moving “head” $y$ until $y \rightarrow x$, ie. “worm closes” $\rightarrow$ new configuration in $Z$
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Global change obtained from sequence of local updates

Each local step gives information on 2-point function

Very close to Adams & Chandrasekharan for $U(N)$
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Local Metropolis, 4³x2 at $\mu_c, m_q = 0.025$

Worm, same parameter set
Consistency check with HMC

Worm–MDP vs. HMC (Forcrand and Kim ’06) \( \beta = 0 \), same volume (\( \mu = T = 0 \))

\( m_{\pi}, \text{HMC} \)
\( \sigma/N_c \)
\( \sigma/N_c, \text{Mean–Field} \)
\( m_{\pi} \)
\( \sigma/N_c, \text{Worm} \)
\( m_{\pi} \)
Consistency check with HMC

Worm-MDP vs. HMC (Forcrand and Kim '06) $\beta = 0$, same volume ($\mu = T = 0$)
Sign problem?

Worst case $m = 0$:

Can reach $\sim 16^3 \times 4 \forall \mu$, ie. adequate
Transition $T = 0, \mu = \mu_c$

Puzzle:
- Mean-field baryon mass is $\approx 3 \Rightarrow$ expect $\mu_c = \frac{1}{3} F_B(T = 0) \approx 1$
- Mean-field estimate $\mu_c \sim 0.55 - 0.66$ much smaller

- Baryon mass $\approx 3$ checked by HMC
- $\mu_c \approx 0.63$ checked by Karsch & Mütter for $T = 1/4$ only

Explanation?
- Problem with $m \rightarrow 0$ or $T \rightarrow 0$ extrapolation of MC data?
- Or nuclear attraction $\sim 1/3$ baryon mass!

Check with $m = 0$, $T \approx 0$ worm simulations
Consistency check with Karsch & Mütter

Baryon number density $n_B$, $8^3 \times 4$, $m_q = 0.1$, Worm vs. Metropolis

Agreement except at $\mu = 0.68 \sim \mu_c$ $\leftrightarrow$ ergodicity of local update
Reducing the quark mass

Baryon number density $n_B$, $L^3 \times 2$ (4), $m_q = 0.1$, Worm vs. Metropolis

As $m \to 0$, $\mu_c$ decreases and transition becomes stronger
Reducing the quark mass

Baryon number density $n_B$, $L^3 \times 2$ (4), $m_q = 0.025$

As $m \rightarrow 0$, $\mu_c$ decreases and transition becomes stronger.
Reducing the quark mass

Baryon number density $n_B$, $L^3 \times 2 (4)$, $m_q = 0$

As $m \rightarrow 0$, $\mu_c$ decreases and transition becomes stronger
Varying the mass at fixed $T = 1/2$

Baryon number density $n_B$, $10^3 x 2$, varying $m_q$

From first-order ($m = 0$) to crossover ($m = 0.1$) $\Rightarrow$ critical mass $m_c$?
Critical mass $m_c(T = 1/2)$?

Histogram $n_B$, $\mu = \mu_c$, $T = 1/2$, $m = 0$

$m = 0.025$

$m = 0.05$

$m = 0.1$

Critical mass $m_c(T = 1/2) \sim 0.05$

Qualitative agreement, but not quantitative
$m = 0$: compare $\mu_c(T = 1/2, T = 1/4)$ with Nishida (2004)

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\( m = 0 \): compare \( \mu_c(T = 1/2, T = 1/4) \) with Kawamoto (2005)

\[
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\]

Take \( T_c = 5/3 \) (mean-field) [MC: 1.40 Karsch]

→ qualitative agreement, but not quantitative
Conclusions

Summary

- For $m = 0$, $\mu_c(T = 1/4) \approx 0.62 (< m_B/3)$ and $\mu_c(T = 1/2) \approx 0.54$
- Critical end-point (not chiral) moves to larger $\mu$ as $m$ increases

Outlook

- Improve systematics:
  Multicanonical MC for first-order transition at low $T$
  Asymmetry $\gamma$ in Dirac coupling to vary $T$ continuously
  Check mean-field “scaling” $T = \gamma^2/N_t$
  Compare real and imaginary $\mu$
- Determine phase diagram:
  Tricritical point for $m = 0$
  Critical end-point as a function of $m$
- Extend to 2 KS fields:
  Baryon no longer self-avoiding $\rightarrow B\pi$ scattering etc..
  Isospin $\mu$